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Parametrizations for Daubechies wavelets

Jean-Marc Lina[1,2*] and Michel Mayrand[2†]
[1]*Atlantic Nuclear Services Ltd., Fredericton, New Brunswick, Canada E3B 5C8*
[2]*Laboratoire de Physique Nucléaire, Université de Montréal, Case Postale 6128, Succursale A, Montréal, Québec, Canada H3C 3J7*
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Two parametrizations are presented for the Daubechies wavelets. The first one is based on the correspondence between the set of multiresolution analysis with compact support orthonormal basis and the group $SU_I(2,\mathbb{C}[z,z^{-1}])$ developed by Pollen. In the second parametrization, emphasis is put on the regularity condition of the Daubechies wavelets and a solitonic cellular automaton algorithm is introduced to solve the orthonormality conditions characterizing the Daubechies wavelets.

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By now, wavelet and multiresolution analysis are well-known tools for the study of physical systems where nonlinearities are dominant [1]. The basic equation of the multiresolution theory is the scaling equation that establishes a connection between the two symmetries underlying the wavelet theory: dilations and translations. We start by briefly calling a few basic results that will be useful in stating the aim of the present work within the proper context.

Given a set of coefficients a_k, $k\in\mathbb{Z}$, the scaling equation

$$\varphi(q)=2\sum_k a_k\varphi(2q-k)\ ,\quad q\in\mathbb{R} \tag{1}$$

and the normalization

$$\int dq\ \varphi(q)=\sum_k a_k=1 \tag{2}$$

define a scaling function $\varphi(q)$. For convenience, we will consider real scaling coefficients in the present work. By defining the set of translations of the dilated function $\varphi(q)$,

$$\varphi_{j,k}(q)=2^{j/2}\varphi(2^jq-k)\ , \tag{3}$$

the multiresolution analysis of $L^2(\mathbb{R})$ consists in the decomposition of the Hilbert space $L^2(\mathbb{R})$ into the chain of closed subspaces

$$\cdots V_{j-1}\subset V_j\subset V_{j+1}\subset\cdots\ , \tag{4}$$

where

$$V_j=\mathrm{Span}\{\varphi_{j,k}(q),\ k\in\mathbb{Z}\} \tag{5}$$

and such that

$$\bigcap_j V_j=\{0\},\quad \bigcup_j V_j=L^2(\mathbb{R})\ . \tag{6}$$

Multiresolution aims to decompose $L^2(\mathbb{R})$ as

$$L^2(\mathbb{R})=V_{j0}\oplus\sum_{j\geq j0}W_j\ , \tag{7}$$

where W_j is defined as the orthogonal complement of V_j in V_{j+1}:

$$V_{j+1}=V_j\oplus W_j\ , \tag{8}$$

and, for a given scale j, W_j is generated by the translations of the dilated wavelet $\psi_j(q)=2^{j/2}\psi(2^jq)$ associated with the multiresolution analysis:

$$\psi(q)=2\sum_k b_k\varphi(2q-k)\ ,\quad q\in\mathbb{R}\ . \tag{9}$$

The choice $b_k=(-1)^k\bar{a}_{1-k}$ insures that $\{\psi_j(q-2^{-j}k),\ k\in\mathbb{Z}\}$ is an orthonormal basis of W_j. We refer the reader to the seminal works of Meyer, Mallat, and Daubechies listed in Ref. [2] for the detailed theory of wavelet and multiresolution analysis. The sets of coefficients $\{a_k\}$ and $\{b_k\}$ form a quadrature mirror filter (QMF). They contain all the information and the properties that characterize discrete wavelet analysis. A particularly important class of QMF is associated with *orthonormal bases of compactly supported wavelets.* The present Rapid Communication aims to describe two ways of classifying that set also known as the Daubechies wavelets family [3].

A necessary condition for the fulfillment of the orthonormality property can be stated as follows. Considering the unit circle in the complex plane, $|z|=1$, we define the entire function $F(z)$ associated with the scaling coefficients:

$$F(z)=\sum_k a_kz^k,\quad \text{with } F(1)=1\ . \tag{10}$$

Orthonormality of the set $\{\varphi(q-n),\ n\in\mathbb{Z}\}$ implies the relation

$$|F(z)|^2+|F(-z)|^2=1\ . \tag{11}$$

In the first approach, we will use the parametrization of Pollen [4] to obtain all the polynomial solutions of Eq. (11). Using Lawton's theorem [5], we will verify that these solutions always define orthonormal bases. In fact, as already found in the original work of Daubechies, the space of solutions of Eq. (11) is highly degenerate and the *Daubechies wavelets* correspond to a small subset defined by the supplementary regularity conditions

$$F'(-1)=F''(-1)=\cdots=F^{(l)}(-1)=0\ . \tag{12}$$

*Electronic address: lina@lps.umontreal.ca
†Electronic address: mayrand@lps.umontreal.ca

The larger l is, the more regular is the wavelet. However, the support of the wavelet also increases. In some sense, the Daubechies wavelets optimize the regularity for a given compact support. It is worth recalling the lemma due to Daubechies that gave the first algorithm for the computation of the scaling coefficients for her wavelet's family. The first part of this lemma states that any entire polynomial (10) solution of Eqs. (11) and (12) is such that

$$|F(z)|^2=[(1+z)/2]^{l+1}[(1+z^{-1})/2]^{l+1} \times \sum_{j=0}^{l} 2^{-2j} \binom{l+j}{j} (1-z)^j(1-z^{-1})^j + \left[\frac{1-z^2}{2}\right]^{l+1} \left[\frac{1-z^{-2}}{2}\right]^{l+1} \mathcal{R}\left[\frac{z^{-1}+z}{2}\right], \qquad (13)$$

where $\mathcal{R}$ is an odd polynomial that obeys some constraint equations that are in general irrelevant since we usually take $\mathcal{R}=0$. In the second part of her lemma, Daubechies uses the spectral factorization theorem to extract $F(z)$ from its modulus given as in Eq. (13). Fixing some phase freedom present in this computation, she exhibits particular solutions for the $2l+2$ scaling coefficients a_k that come out from Eq. (13). Frequently used in the applications of discrete wavelets, those *maximal* phase solutions (also called "least symmetric" Daubechies wavelets) are the cases investigated here. The two different parametrizations discussed here characterize either the nonlinear constraints (11) or the linear conditions (12) on the scaling coefficients. We then identify the Daubechies wavelets in each parameter space which happens to be of the same dimensionality equal to l. The main result in the second approach is the derivation of a solitonic cellular automaton for describing the scaling coefficients of the Daubechies wavelets family.

The parametrization of Pollen relies on the correspondence between the orthonormal multiresolution analysis and the elements of the group $SU_I(2,\mathbb{C}[z,z^{-1}])$. The elements of this group are the unitary two-by-two matrices with components in $\mathbb{C}[z,z^{-1}]$ and unimodular at $z=1$. Let us recall his analysis. Considering the group of the two-by-two matrices of the form

$$g(z)=\begin{bmatrix} u(z) & v(z) \\ -\bar{v}(z) & \bar{u}(z) \end{bmatrix}, \qquad (14)$$

where $u(z)$ and $v(z)$ are polynomials in $\mathbb{C}[z,z^{-1}]$ and z is constrained on the unit circle, Pollen noticed that

$$F(z)=\mathbf{c}(1)\cdot g(z^2)\mathbf{c}(z), \qquad (15)$$

where $\mathbf{c}(z)$ is the unit vector

$$\mathbf{c}(z)=\frac{1}{\sqrt{2}}\begin{bmatrix} z \\ -1 \end{bmatrix}, \qquad (16)$$

solves Eq. (11) if and only if $g(z)$ is unitary. This observation led Pollen to investigate the group $SU(2,\mathbb{C}[z,z^{-1}])$ and more precisely $SU_I(2,\mathbb{C}[z,z^{-1}])$ since we force $g(1)$ to be the identity matrix. This takes into account the condition $F(1)=1$.

Besides this correspondence between orthonormal multiresolution analysis and the group $SU_I(2,\mathbb{C}[z,z^{-1}])$, Pollen established the *unique factorization theorem, which states that any element of this* group can be factorized in a unique way:

$$g(z)=U_1(z)U_2(z)\cdots U_m(z), \qquad (17)$$

with

$$U_i(z)=g_{2i-1}(z)g^{\dagger}_{2i}(z), \quad i=1,2,3,\ldots. \qquad (18)$$

The $g_i(z)$ are elements labeled with two angles:

$$g_i(z)=g_{\theta_i,\varphi_i}(z)=\begin{bmatrix} u_{\theta_i,\varphi_i}(z) & v_{\theta_i,\varphi_i}(z) \\ -\bar{v}_{\theta_i,\varphi_i}(z) & \bar{u}_{\theta_i,\varphi_i}(z) \end{bmatrix}, \qquad (19)$$

with

$$u_{\theta_i,\varphi_i}(z)=\sin^2(\theta_i)z^{-1}+\cos^2(\theta_i), \quad v_{\theta_i,\varphi_i}(z)=\sin(\theta_i)\cos(\theta_i)e^{i\varphi_i}(z-1). \qquad (20)$$

Defining $\nu=\tan(\theta)$, we obtain

$$g_{\nu,\varphi}(z)=\begin{bmatrix} u_{\nu,\varphi}(z) & v_{\nu,\varphi}(z) \\ -\bar{v}_{\nu,\varphi}(z) & \bar{u}_{\nu,\varphi}(z) \end{bmatrix}, \text{ with } u_{\nu,\varphi}(z)=1+[\nu^2/(1+\nu^2)](z^{-1}-1), \quad v_{\nu,\varphi}(z)=[\nu/(1+\nu^2)](z-1)e^{i\varphi}. \qquad (21)$$

The number of g's and $g^{\dagger}$'s in the product (17) is denoted by J and is called the *order* of the underlying analysis. For example, the wavelet of order 1 ($J=1$) admits four nonvanishing scaling coefficients associated with $g_{\nu,\varphi}$:

$$a_{-1}=[\nu(\nu+e^{i\varphi})]/2(1+\nu^2), \quad a_0=[(1+e^{-i\varphi}\nu)]/2(1+\nu^2), \quad a_1=(1-e^{i\varphi}\nu)/2(1+\nu^2), \quad a_2=[\nu(\nu-e^{-i\varphi})]/2(1+\nu^2). \qquad (22)$$

When $\varphi=0$, we recognize here a family of coefficients already mentioned in the early works of Daubechies [2]. We restrict our attention to the interval $|\nu|\le 1$ in which the Haar analysis is found at the group identity ($\nu=0$) and for $|\nu|=1$.

The factorization (17) leads to an iterative algorithm for the computation of the scaling coefficients from a set of parameters $\{\nu_i, i=1,\ldots,J\}$. Starting with the Haar coefficients $a_k^{(0)}=a_k(\nu=0)$, we have the following chain rule:

$$a_k^{(j)}=[1/(1+\nu_j^2)][\bar{a}_k^{(j-1)}+(-1)^j\nu_j e^{i(-1)^{j+k}\varphi_j}[\bar{a}_{k-1}^{(j-1)}-\bar{a}_{k+1}^{(j-1)}]+\nu_j^2 a_{k+2(-1)^j+k}^{(j-1)}], \qquad (23)$$

ending with the set of a_k equal to $a_k^{(J)}$, $k=-J,-J+1,\cdots,J,J+1$. Let us notice that the number of nonvanishing coefficients grows with j from 2 (Haar) to $2J+2$. Since this work is concerned with real scaling coefficients, we set $\varphi_i=0\ \forall\ i$.

To check the orthornormality of the multiresolution analysis described by the coefficients (22), we can use a lemma of Lawton [5] that relates this property to the spectrum of the operator A defined as

$$A_{i,j}=\sum_k a_{j-2i+k}a_k\ , \qquad i,j=-2J-1,-2J,\ldots,2J,2J+1\ . \tag{24}$$

Lawton has shown that orthornormality was equivalent to the presence of a nondegenerate eigenvalue equal to $\frac{1}{2}$. Such is the case with the coefficients (22), for any ν. Moreover, Pollen's *factorization theorem* extends this result to any multiresolution analysis parametrized by a set of ν_i.

Before implementing the second set of equations (12), in order to compute the ν_i associated with the Daubechies wavelets, we notice some interesting symmetry properties of the scaling coefficients:

$$\{\nu_i\to-\nu_i\}\ \text{then}\ \{a_k\to a_{1-k}\}\ ,$$
$$\left\{\nu_i\to\frac{1}{\nu_i}\right\}\ \text{then}\ \{a_k\to a_{k+(-1)^{J+k}}\}\ ,\quad \forall\ i,k\ . \tag{25}$$

These symmetries will be preserved only for the Haar cases when we consider the set of solutions of the linear equations (12).

Since we have J parameters we can impose $l=J$, Eqs. (12). We have carried out the computations numerically (only cases $J=1$ and 2 can be solved algebraically) and, restricting the parameters to the domain $|\nu|\le 1$, we have found the values plotted in Fig. 1. As the order J is increasing, all the parameters ν associated with the Daubechies wavelets start at $\nu=0$ (Haar) and converge towards ±1, i.e., the Haar case. From the point of view of the algorithms for wavelet decomposition which are factorizable in the same way as (17), this observation could reduce the computations involved with high-order multiresolution analysis. Despite the interesting behavior of the parameters [6] displayed in Fig. 1, we must admit the absence of a closed expression for them.

The previous numerical results had suggested the following change of variables on the scaling coefficients. It is the second parametrization proposed in this work:

$$a_k=\frac{1}{2^{J+1}}\sum_{j=0}^{J}M_{k,j}\rho_j\ ,\quad k=-J,-j+1,\ldots,J+1 \tag{26}$$

where M is a $2(J+1)\times(J+1)$ matrix of integers verifying the following equations:

$$M_{-J,j}=1,\quad M_{k,0}=\begin{bmatrix}2J+1\\ J+k\end{bmatrix}\ ,$$
$$M_{k,j}=M_{k,j-1}-M_{k-1,j}-M_{k-1,j-1}\ . \tag{27}$$

The solution is

$$M_{k,j}=\begin{cases}m_{k,j}, & k=1,\ldots,J+1\\ (-1)^j m_{1-k,j}, & k=-J,\ldots,0\end{cases} \tag{28}$$

with

$$m_{k,j}=\sum_{l=0}^{j}(-2)^l\begin{bmatrix}j\\ l\end{bmatrix}\begin{bmatrix}2J+1-l\\ J+k-l\end{bmatrix}\ ,\qquad k=1,\ldots,J+1\ ,\ j=0,1,\ldots J\ . \tag{29}$$

Using simple identities involving binomials, we can easily prove that expression (26) satisfies the J linear equations (12) for any value for the J parameters $\rho_j, j=1,\ldots,J$. The value $\rho_0=2^{-J}$ is fixed by the normalization of the scaling equation, $F(1)=1$. The change of variable (26) is invertible using half of the a_k's:

$$\rho_j=2^{J+1}\sum_{k=1}^{J+1}m_{j,k}^{-1}a_k=2^{J+1}(-1)^j\sum_{k=1}^{J+1}m_{j,k}^{-1}a_{1-k}\ . \tag{30}$$

Surprisingly enough, this relationship recalls the underlying symmetry between a_k and a_{1-k} previously noticed in Eq. (25).

We thus recover a J-fold parametrization of a multiresolution analysis of order J which makes possible the factorization of $F(z)$:

$$F(z)=[(1+z)/2]^{J+1}p(z^{-1})\ , \tag{31}$$

with [7]

$$p(z)=\sum_{j=0}^{J}(z+1)^{J-j}(z-1)^j\rho_j\ . \tag{32}$$

The parametrization (26) is equally interesting because of the regularity conditions contained in (26) that make the linear equations (12) automatically satisfied.

What is left is the computation of the parameters ρ_j that define the Daubechies wavelets. This amounts to solving the quadratic equation (11), and for this purpose we have considered a $1+1$ cellular automation.

Let us consider a one-dimensional infinite mesh with the variable $\rho_j(t)\ge 0$ attached to the nodes j. The time t is discrete and we take $\rho_j(0)=0\ \forall\ j$ as the initial condition. Given $J\ge 0$, we distribute "masses" on the mesh according to the following definition:

$$m_j=\begin{cases}0 & \text{if } j<0 \text{ or} j>J\ ,\\ \dfrac{1}{2^J}\left[\begin{bmatrix}2J+1\\ J\end{bmatrix}\right]^{1/2} & \text{if } 0\le j\le J\ ,\end{cases} \tag{33}$$

and we define the "energy" $\mathcal{E}$ of this system through

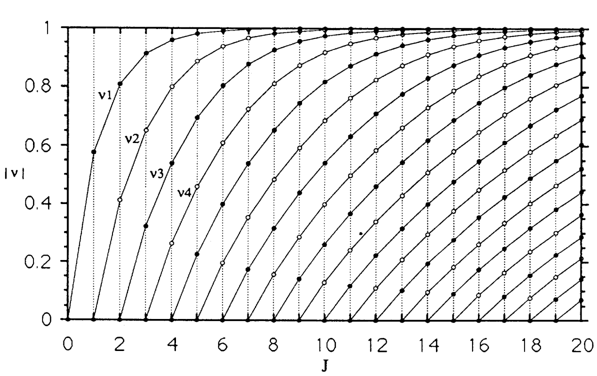

FIG. 1. Parameters ν for the Daubechies wavelets [the sign of ν_i is $(-1)^{i+1}$].

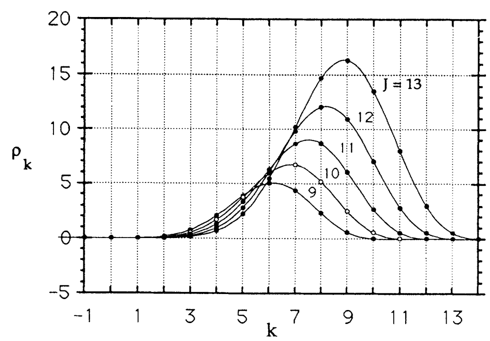

FIG. 2. $\rho(\infty)$ for $J=9,10,11,12,13$.

$\mathcal{E}^2=\frac{1}{2}\sum_j m_j^2$. We have $\mathcal{E}=\frac{1}{2}$ for all J. The cellular automation is now defined for any J by the following equation:

$$\rho_j(t+1)^2=m_j^2-2\sum_{k=1}^{\infty}(-1)^k\rho_{j-k}(t+1)\rho_{j+k}(t)\ ,$$

$$\rho_j(t+1)\geq 0\ . \qquad (34)$$

Because of the time dependence in the right-hand side of (34), this automation does not correspond to the standard definition of Wolfram's cellular automata but rather belongs to the *solitonic cellular automata* family introduced by Park and Steiglitz [8]. We checked numerically that for all the values of J listed in Fig. 1, this cellular automation was converging toward a stable state $\rho(\infty)$ localized on the set of nodes where the masses were defined. Furthermore, the values $\rho_j(\infty)$ were shown to correspond exactly to the Daubechies scaling coefficients, up to the linear transformation (26). Figure 2 exhibits the $\rho(\infty)$ configurations for $J=9$ to 13.

In order to characterize the convergence of the automaton, we define its "energy" by

$$\mathcal{E}(t)^2=\sum_n\left|\sum_k a_{k+2n}(t)a_k(t)\right|^2\ , \qquad (35)$$

where the a_k's are given by (26) at each time step. The criterion for the convergence towards a stable state is defined as the limit of $\mathcal{E}(t)$ equal to the energy of the system, i.e., $\frac{1}{2}$. As J increases, this convergence requires more and more time steps. Let us notice that we could have used the Lawton operator (24) to define the energy as $\frac{1}{4}\,\mathrm{Tr}[A(t)]$. Right at the beginning, transients and oscillations take place during a "relaxation" time t_r. This transient regime, of small duration for low values of J, can be very long for higher order, and metastable states can even occur during this period. Figure 3 shows the case $J=20$. Numerical simulations, however, tend to demonstrate that a unique stable state is always reached after a finite time [6]. It corresponds to the Daubechies scaling coefficients.

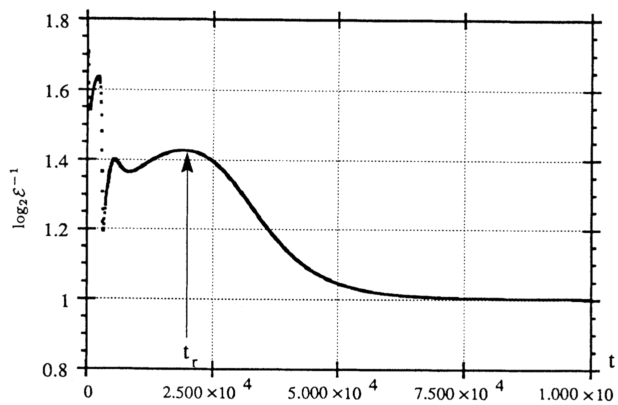

FIG. 3. $\log_2\mathcal{E}^{-1}$ vs t for $J=20$.

In view of the growing relevance of multiresolution analysis for the investigation of physical phenomena, the classification of the discrete wavelets on the basis of their properties (compactness of the support, regularity, orthonormality) becomes an important matter. Looking at the compactly supported wavelets of Daubechies, we first have used the Pollen parametrization and we have identified the *least symmetric* Daubechies wavelets at different order in the domain $|\nu|\leq 1$. An open problem is to obtain the more symmetrical wavelets in this parameter space, possibly outside $|\nu|\leq 1$. The second parametrization introduced in (26) has exhibited a typical nonlinear physical system with the *solitonic cellular automaton (34)*. It is worth mentioning that numerical computations have shown that the set of equations (34) at $t=\infty$, i.e.,

$$\rho_j^2=m_j^2-2\sum_{k=1}^{\mathrm{Min}[j,J-j]}(-1)^k\rho_{j-k}\rho_{j+k}\ ,\quad \forall_j\ , \qquad (36)$$

are *equivalent* to the nonlinear conditions (11) and thus are satisfied by any Daubechies scaling coefficient. However, it happens that only those that correspond to the *least symmetric* ones are obtainable with a cellular automaton with, and only with, a "solitonic" interacting term. Needless to say, the cellular automaton (34) by itself deserves a more complete analysis in light of the recent works about integrable cellular automata [9].

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[1] A. Arneodo *et al.*, Phys. Rev. Lett. **61**, 2281 (1988); D. Permann and I. Hamilton, *ibid.* **69**, 2607 (1992); J. Liandrat and F. Moret-Bailly, Eur. J. Mech. B **9**, 19 (1990); M. Farge, Ann. Rev. Fluid. Mech. **24**, 395 (1992).

[2] Y. Meyer, *Ondelettes* (Hermann, Paris, 1991); I. Daubechies, *Ten Lectures on Wavelets,* SIAM, CBMS Series (Publisher, City, 1992); S. Mallat, IEEE Trans. Pattern Anal. Mach. Intell. **11**, 674 (1989).

[3] I. Daubechies, Comm. Pure Appl. Math. **41**, 909 (1988).

[4] D. Pollen, Am. Math. Soc. **3**, 611 (1990).

[5] W. Lawton, J. Math. Phys. **32**, 57 (1991).

[6] The numerical computations were carried out with MATHEMATICA. Programs and numerical results are made available via ftp anonymous at lpsvsh.lps.umontreal.ca in the WAVELET directory.

[7] During the completion of this work, we found a similar result in A. N. Akansu, R. A. Haddad, and H. Caglar, IEEE Trans. Signal Proc. **41**, 13 (1993).

[8] J. K. Park and K. Steiglitz, Physica **19D**, 333 (1986).

[9] M. Bruschi, P. M. Santini, and O. Ragnisco (unpublished).